MATH 2850: 5.1 - LINEAR SECOND ORDER ODEs

DETERMINANTS OF 2×2 **MATRICES:** For a 2×2 matrix,

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

EXAMPLE: $\begin{vmatrix} 4 & -3 \\ 2 & 1 \end{vmatrix} = (4)(1) - (-3)(2) = 4 + 6 = 10$

CRAMER'S RULE: The system of linear equations: $\begin{cases} ax + by &= e \\ cx + dy &= f \end{cases}$ has a unique solution provided

$$D = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc \neq 0$$

In this case, the solution to the system can be found as follows: letting $D_1 = \begin{vmatrix} e & b \\ f & d \end{vmatrix}$ and $D_2 = \begin{vmatrix} a & e \\ c & f \end{vmatrix}$,

$$x = \frac{D_1}{D}, \quad y = \frac{D_2}{D}$$

EXAMPLE: Use Cramer's Rule to solve: $\begin{cases} 3x + 7y &= 5 \\ 2x + 12y &= -2 \end{cases}$

Ans: $D = 22 \neq 0$, $D_1 = 74$, and $D_2 = -16$ so $x = \dfrac{74}{22} = \dfrac{37}{11}$ and $y = \dfrac{-16}{22} = -\dfrac{8}{11}$

RECALL: The IVP: $y' + p(x)y = f(x)$, $y(x_0) = y_0$ has a unique solution provided the functions p, f are continuous on an open interval containing x_0. We now consider the second order linear IVP:

$$y'' + p(x)y' + q(x)y = f(x), \; y(x_0) = k_0, \; y'(x_0) = k_1$$

Note here we have two IC's: one for the function y and one for the derivative y'. To be an IVP, it's important to realize we have information about y and y' at the **same** point, in this case x_0.

It seems reasonable we need two conditions to pin down a solution to a second order ODE since, ostensibly, we have two 'integrations' worth of constants to determine.

As long as p, q, and f are continuous in an open interval containing x_0, we are guaranteed a unique solution!

RECALL: A linear ODE is called **homogeneous** if $f(x) = 0$. That is, $y'' + p(x)y' + q(x)y = 0$.

NOTE: The function $y(x) = 0$, called the **trivial solution** is **always** a solution to a homogeneous linear ODE.

Moreover, the trivial solution is also a solution to the IVP: $y'' + p(x)y' + q(x)y = 0$, $y(x_0) = 0$, $y'(x_0) = 0$.

Because of the **uniqueness** of the guaranteed solutions to linear IVPs, $y(x) = 0$ is the **only** solution to this IVP.

EXAMPLE: Consider the DE: $y'' - y' - 12y = 0$.

- Show $y_1 = e^{-3x}$ and $y_2 = e^{4x}$ are both nontrivial solutions to this DE.

- Show $y = c_1 e^{-3x} + c_2 e^{4x}$ is also a solution to the DE for any choice of constants c_1 and c_2.

- Find c_1 and c_2 so that $y = c_1 e^{-3x} + c_2 e^{4x}$ solves the IVP $y'' - y' - 12y = 0$ subject to:

  $y(0) = 2$, $y'(0) = -1$.

  Ans: $y = \dfrac{9}{7}e^{-3x} + \dfrac{5}{7}e^{4x}$

  $y(1) = 2$, $y'(1) = -3$

  Ans: $y = \dfrac{11e^{3}}{7}e^{-3x} + \dfrac{3e^{-4}}{7}e^{4x}$

EXAMPLE: Prove $y = c_1 e^{-3x} + c_2 e^{4x}$ is the general solution to $y'' - y' - 12y = 0$, $y(x_0) = k_0$, $y'(x_0) = k_1$.

DEFINITION: A **linear combination** of y_1 and y_2 is a function $y = c_1 y_1 + c_2 y_2$ where c_1 and c_2 are constants.

THE SUPERPOSITION PRINCIPLE:

If y_1 and y_2 are solutions to $y'' + p(x)y' + q(x)y = 0$, then so is any linear combination $y = c_1 y_1 + c_2 y_2$.

PROOF:

DEFINITION: A **fundamental set** of solutions to $y'' + p(x)y' + q(x)y = 0$ is a set of functions $\{y_1, y_2\}$ so that the linear combinations of y_1 and y_2, $y = c_1 y_1 + c_2 y_2$, form the **general solution** to the DE.

GOAL: Given a second order ODE, find a fundamental set of solutions so that we can form the general solution.

Suppose $y = c_1 y_1 + c_2 y_2$ is the general solution to the IVP $y'' + p(x)y' + q(x)y = 0$, $y(x_0) = k_0$, $y'(x_0) = k_1$.

Here we assume p and q are continuous on an open interval (a, b).

Then we need to be able to find constants c_1 and c_2 so that:

$$\begin{cases} c_1 y_1(x_0) + c_2 y_2(x_0) &= k_0 \\ c_1 y_1'(x_0) + c_2 y_2'(x_0) &= k_1 \end{cases}$$

for any choice of x_0 in (a, b) and any choice of constants k_0 and k_1.

From Cramer's Rule, we know this system has a unique solution if

$$\begin{vmatrix} y_1(x_0) & y_2(x_0) \\ y_1'(x_0) & y_2'(x_0) \end{vmatrix} = y_1(x_0)y_2'(x_0) - y_2(x_0)y_1'(x_0) \neq 0$$

This gives rise to the notion of the Wronskian.

THE WRONSKIAN: If y_1 and y_2 are differentiable functions, we define $W(x) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1 y_2' - y_2 y_1'$

EXAMPLE: Find the Wronskian of the following sets of functions:

- $\left\{e^{-3x}, e^{4x}\right\}$

Ans: $W(x) = 7e^{x}$

- $\left\{e^{2x}, xe^{2x}\right\}$

Ans: $W(x) = e^{4x}$

THEOREM: If y_1, y_2 are solutions to $y'' + p(x)y' + q(x)y = 0$ in an open interval (a, b), then:

- The Wronskian $W(x)$ is either never zero or always zero.

- If $W(x_0) \neq 0$ for some x_0 in (a, b) then $\{y_1, y_2\}$ form a fundamental set of solutions and vice-versa.

We'll prove this theorem later - for now we apply this to a given IVP.

EXAMPLE: Consider the IVP: $x^2y'' - 2xy' - 10y = 0$, $y(-1) = 3$, $y'(-1) = 2$.

- Show the IVP will have a unique solution on $(-\infty, 0)$.

- Show $\left\{x^{-2}, x^5\right\}$ is a fundamental set of solutions to this DE.

- Solve the IVP.

Ans: $y = \dfrac{17}{7}x^{-2} - \dfrac{4}{7}x^5$

LINEAR INDEPENDENCE AND THE WRONSKIAN

DEFINITION:

A set of functions $\{y_1, y_2\}$ is called **linearly independent** if the only solution to $c_1y_1 + c_2y_2 = 0$ is $c_1 = c_2 = 0$.

A set of functions is $\{y_1, y_2\}$ is called **linearly dependent** if it is not linearly independent.
That is, there are constants c_1 and c_2 both not zero where $c_1y_1 + c_2y_2 = 0$.

EXAMPLE:

- Show the set $\{x, 3x\}$ is linearly dependent.

- Show the set $\left\{x, x^2\right\}$ is linearly independent.

NOTE: The set $\{y_1, y_2\}$ is linearly independent if and only if y_1 and y_2 aren't multiples of each other.

THEOREM: Consider the set of functions $\{y_1, y_2\}$ on an interval (a, b).

If $W(x_0) \neq 0$ for some x_0 in (a, b), then $\{y_1, y_2\}$ is linearly independent on (a, b).

PROOF:

NOTE: The set $\left\{x^4, x^3|x|\right\}$ is linearly independent despite $W(x) = 0$ for all x. (Can you show this?)

ABEL'S FORMULA

Suppose y_1 and y_2 are solutions to $y'' + p(x)y' + q(x)y = 0$ where p and q are continuous on an interval (a, b).

Let W be the Wronskian of $\{y_1, y_2\}$. If x_0 is any point in (a, b), then

$$W'(x) = -p(x)W(x) \implies W(x) = W(x_0)\, e^{-\int_{x_0}^{x} p(t)\, dt}$$

Hence, $W(x)$ is either never 0 on (a, b) or $W(x) = 0$ for all x on (a, b).

PROOF:

PUTTING IT ALL TOGETHER: Suppose p and q are continuous on an open interval (a, b) containing x_0.

If y_1 and y_2 are solutions to $y'' + p(x)y' + q(x)y = 0$ then the following are equivalent:

- the Wronskian $W(x_0) \neq 0$
- the Wronksian $W(x) \neq 0$ for all x in (a, b).
- $\{y_1, y_2\}$ is linearly independent.
- $\{y_1, y_2\}$ is a fundamental set of solutions to $y'' + p(x)y' + q(x)y = 0$.
- $y = c_1y_1 + c_2y_2$ is the general solution to $y'' + p(x)y' + q(x)y = 0$.

HOMEWORK: Pg. 203: 1-13 odd